Quick Summary

$$mu'' + \gamma u' + ku = F(t)$$

Case 1: F(t) = 0 and $\gamma = 0$

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t)$$

$$\omega_0 = \sqrt{k/m} = \boldsymbol{natural}\ freq.$$

Case 2: F(t) = 0 and $\gamma > 0$

2a: $\gamma > 2\sqrt{mk}$, **overdamped**

2b: $\gamma = 2\sqrt{mk}$, **critically damped**

2c: $\gamma < 2\sqrt{mk}$, **damped vibrations**

$$u(t) = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$$

$$\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}} = \text{ } \textbf{quasi}\text{-frequency}$$

Case 3: $F(t) = F_0\cos(\omega t)\ and\ \gamma = 0$

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + u_p(t)$$

3a: $\omega \neq \omega_0$: $u_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$

3b: $\omega = \omega_0$: $u_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega t)$

Resonance!

Case 4: $F(t) = F_0\cos(\omega t)\ and\ \gamma > 0$

Sol'n: $u(t) = u_c(t) + u_p(t)$

$u_c(t)$ =homogeneous sol'n = ***transient*** sol'n

$u_p(t)$ =particular sol'n = ***steady state*** sol'n

(also called ***forced response***)

We will discuss Case 4 today.

Here is an example:

Entry Task:

m = 1 kg, γ = 2 N/(m/s), k = 5 N/m.

External Forcing: F_0 = 10 N, $\omega = 1$ rad/s.

1. Find the homogenous solution.
2. Find a particular solution.

The general solution to

$$u'' + 2u' + 5u = 10\cos(t)$$

is

u(t) =

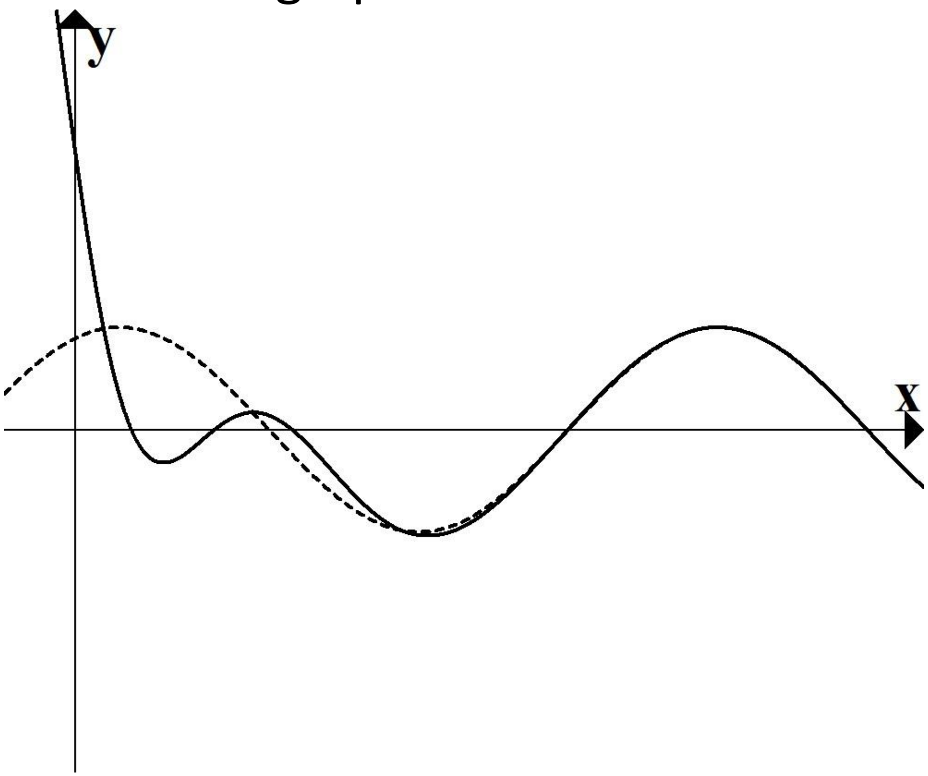
$$e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + 2\cos(t) + \sin(t)$$

If you are given initial conditions

$u(0) = 6$ and $u'(0) = -11$,

then you can find $c_1 = 4$ and $c_2 = -6$.

Here is the graph of that solution:



Example: Same problem as before with smaller damping:

m = 1 kg, γ = 0.1 N/(m/s), k = 5 N/m.

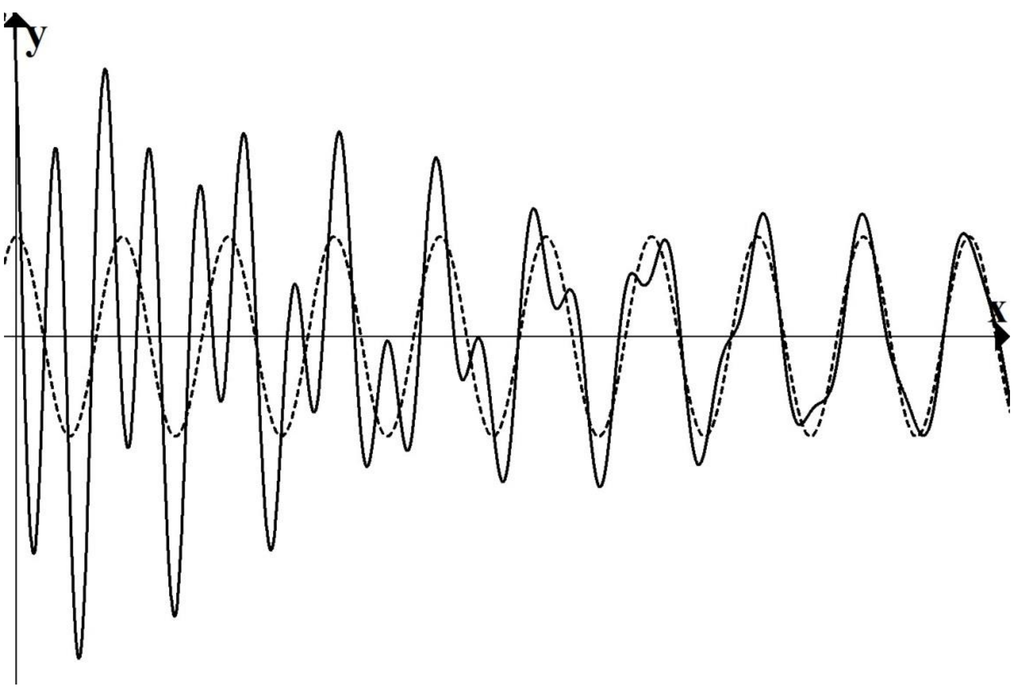
External Forcing: F_0 = 10 N, $\omega = 1$ rad/s.

$$u'' + 0.1u' + 5u = 10\cos(t)$$

Solution:

$$\mathrm{u(t)} = e^{-0.05t}(c_1\cos(\mu t) + c_2\sin(\mu t)) + 2.4984\cos(t) + 0.0624\sin(t)$$

Here is the graph with c_1 = 4 and c_2 = -6.



Some First Observations:

1. When there is damping, there is a *transient* part of the solution that always dies out.
2. If damping is smaller, it takes longer to die out.
3. The amplitude of the *steady state* solution is dependent on m, γ, k, and F_0 in some way.

Consider the example

$$u'' + \gamma u' + 5u = 10\cos(\omega t)$$

Homogenous Solution:

The characteristic equation is

$r^2 + \gamma r + 5 = 0$

$r = -\frac{\gamma}{2} \pm \frac{1}{2}\sqrt{\gamma^2 - 20}$

If $\gamma < \sqrt{20}$, then

$$\mu = \frac{1}{2}\sqrt{20 - \gamma^2} = \sqrt{5 - \frac{\gamma^2}{4}}$$

Note: $w_0 = \sqrt{5}$

Particular Solution:

Using: $u_p(t) = A\cos(\omega t) + B\sin(\omega t)$

we get

$$(5 - \omega^2)A + \gamma\omega B = 10$$
$$-\gamma\omega A + (5 - \omega^2)B = 0$$

$$A = \frac{10(5 - \omega^2)}{\omega^4 + (\gamma^2 - 10)\omega^2 + 25}$$
$$B = \frac{10\gamma\omega}{\omega^4 + (\gamma^2 - 10)\omega^2 + 25}$$

as you can see it starts to get messy.

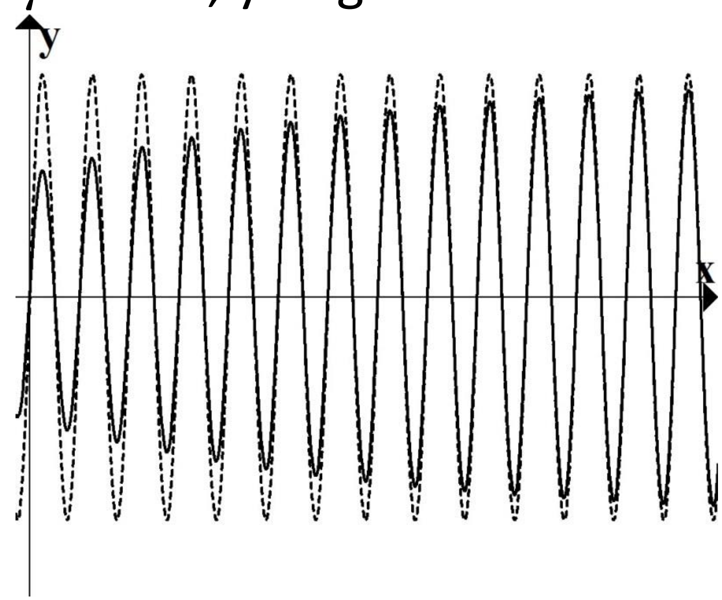
Let's look at the case when

$$\omega = \sqrt{5}$$

then A = 0 and $B = \frac{10\gamma\sqrt{5}}{25+(\gamma^2-10)5+25} = \frac{2\sqrt{5}}{\gamma}$

$$u_p(t) = \frac{2\sqrt{5}}{\gamma}\sin(\sqrt{5}t)$$

When $\gamma = 0.1$, you get



$$u'' + \gamma u' + 5u = 10\cos(\sqrt{5}\, t)$$

Steady state solution:

$$u_p(t) = \frac{2\sqrt{5}}{\gamma}\sin(\sqrt{5}\, t)$$

γ	R
10	0.447
1	4.47
0.1	44.72
0.01	447.21
0.001	4472.14

Some Second Observations:

1. If the forcing frequency is close to the natural frequency, then tend to get large amplitude solutions.
2. In this case, the amplitude gets larger and larger the closer the damping is to zero.

General Discussion

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t)$$

Note: $\omega_0 = \sqrt{\frac{k}{m}}$, $\mu = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$

Particular Solution:

$$u_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

Leads to

$$-\gamma\omega A + (k - m\omega^2)B = 0$$

$$(k - m\omega^2)A + \gamma\omega B = \mathrm{F}_0$$

The formulas for A and B are large to write out.

The amplitude of the steady state solution simplifies to:

$$R = \sqrt{A^2 + B^2} = \frac{F_0}{\sqrt{(k-m\omega^2)^2+\gamma^2\omega^2}}$$

Thinking of this as a function of ω the maximum steady state amplitude occurs when:

$$\omega_{max} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{2m^2}}$$

In particular, for small values of γ if $\omega \approx \omega_0$, then $R \approx \frac{F_0}{\gamma\omega}$ is large.

(resonance)